Random Iteration Algorithm for Graph-Directed Sets

Yoshiki Tsujii[1], Takakazu Mori[1], Mariko Yasugi[2], and Hideki Tsuiki[3]

[1] Faculty of Science, Kyoto Sangyo University
tsujiiy,morita@cc.kyoto-su.ac.jp
[2] Kyoto Sangyo University
yasugi@cc.kyoto-su.ac.jp
[3] Graduate School of Human and Environmental Studies, Kyoto University
tsuiki@i.h.kyoto-u.ac.jp

Abstract. A random iteration algorithm for graph-directed sets is defined and discussed. Similarly to the Barnsley-Elton's theorem, it is shown that almost all sequences obtained by this algorithm reflect a probability measure which is invariant with respect to the system of contractions with probabilities.

1 Introduction

The motif of this article is the random iteration algorithm for a family of graph-directed sets. According to Barnsley [1], the random iteration algorithm can be used to picture a fractal defined by a finite number of contractions. Our interest is to extend this idea to graph-directed sets (cf. [7], [8], [9], [10]).

Our present interest was originally motivated by the work of Brattka [4], in which Brattka presented an example of a "Fine-computable" function which is not "locally uniformly Fine-computable." The graph of Brattka's function can be characterized as a graph-directed set, and in [10] we have depicted graphs of some graph-directed sets by using a deterministic algorithm.

The random iteration algorithm is an alternative for picturing some invariant sets. Let us briefly explain this algorithm according to Barnsley and Elton (cf. [1], [2], [6]).

Let $\{S_1, S_2, \ldots, S_K\}$ be a family of contractions on $\mathbf{R}^d$. Let $(p_1, p_2, \ldots, p_K)$ be a system of probabilities assigned to $\{S_1, S_2, \ldots, S_K\}$, where $p_i > 0$ ($i = 1, \ldots, K$) and $\sum_{i=1}^{K} p_i = 1$. Choose $x(0) \in \mathbf{R}^d$ and choose randomly, recursively and independently $x(t) \in \{S_1(x(t-1)), S_2(x(t-1)), \ldots, S_K(x(t-1))\}$, where the probability for the event $x(t) = S_i(x(t-1))$ is p_i. The sequence $\{x(0), x(1), \ldots, x(n), \ldots\}$ "converges to" the invariant set with respect to $\{S_1, S_2, \ldots, S_K\}$. Moreover, the density of points in this sequence reflects a measure which is invariant with respect to $\{S_1, S_2, \ldots, S_K\}$ and $(p_1, p_2, \ldots, p_K)$ in the sense of Theorem 2 (Barnsley and Elton). Let us give an example.

Example 1 (Koch Curve). The Koch curve is invariant for S_1, S_2, S_3, S_4, where S_i maps the whole triangle to a smaller triangle for $i = 1, 2, 3, 4$ (cf. Fig. 1).

Andrej Bauer, Peter Hertling, Ker-I Ko (Eds.)
6th Int'l Conf. on Computability and Complexity in Analysis, 2009, pp. 245-256
http://drops.dagstuhl.de/opus/volltexte/2009/2275

Let $(3/7, 1/7, 2/7, 1/7)$ be a system of probabilities assigned to $\{S_1, S_2, S_3, S_4\}$. Starting with $x(0) = (0, 0)$, we obtained the figure after 4000 times loop.

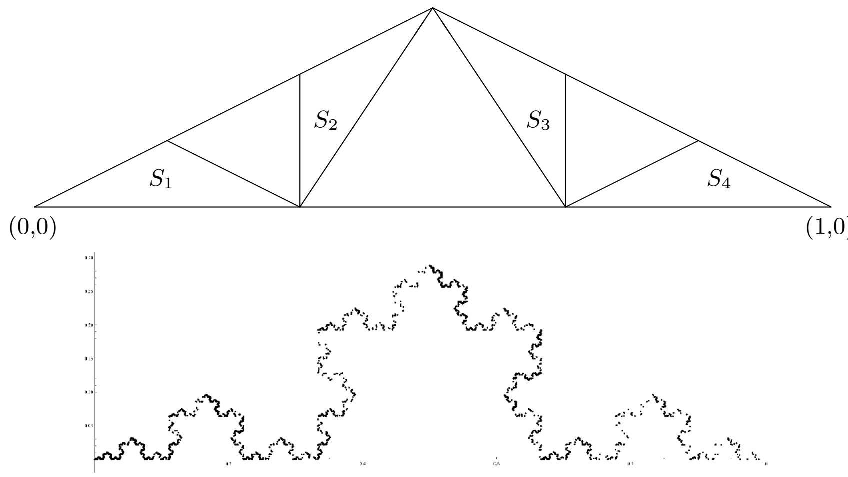

Fig. 1. Koch curve drawn with the random iteration algorithm.

In Section 2, we review the theory of graph-directed sets, and then explain the random iteration algorithm for graph-directed sets. In Section 3, we prove the Barnsley-Elton theorem for graph-directed sets (Theorems 3-5 and Corollary 1). At the end, another random iteration algorithm is proposed and some results thereof are previewed; details will be developed later.

We might note that I. Werner has investigated a random iteration algorithm for a family of graph-directed sets in a different approach in [11].

2 Random iteration algorithm for graph-directed sets

Graph-directed sets are defined as follows ([3], [5] and [9]). Let $K \geq 2$. Let $V = \{1, \ldots, K\}$ be a set of vertices, and let $E_{k,l}$ be a set of edges from vertex l to vertex k. Put $E = \{E_{k,l}\}_{k,l \in V}$. Assume that $\cup_{l=1}^{K} E_{k,l} \neq \emptyset$ for each k, although some of $E_{k,l}$'s may be empty. Let $E_{i,j}^k$ be the set of sequences of k edges $(e_1, e_2, \ldots, e_k)$ which is a directed path from vertex j to vertex i. We say that the graph is transitive if, for any i, j, there is a positive integer p such that $E_{i,j}^p$ is non-empty.

Definition 1 (Graph-directed sets). *Let (V, E) be a transitive directed graph. For each $e \in E_{k,l}$, let S_e be a contraction on a compact space. A K-tuple of non-empty compact sets $(F_1, F_2, \ldots, F_K)$ is called a family of graph-directed sets if it* *satisfies*

$$F_k = \bigcup_{l=1}^{K} \bigcup_{e \in E_{k,l}} S_e(F_l) \quad (k = 1, \ldots, K).$$

If we put

$$\{S_e : e \in E_{k,l}\} = \{S_i^{kl} : i = 1, \ldots, n_{kl}\} \quad (k, l = 1, \ldots, K),$$

the definition above can be stated in the following form.

Definition 2. *Put*

$$\mathcal{S} = \begin{pmatrix} \{S_i^{11}\}_{i=1}^{n_{11}} & \{S_i^{12}\}_{i=1}^{n_{12}} & \ldots & \{S_i^{1K}\}_{i=1}^{n_{1K}} \\ \ldots & \ldots & \ldots & \ldots \\ \{S_i^{K1}\}_{i=1}^{n_{K1}} & \{S_i^{K2}\}_{i=1}^{n_{K2}} & \ldots & \{S_i^{KK}\}_{i=1}^{n_{KK}} \end{pmatrix},$$

where each S_i^{kl} is a contraction on a compact space, $n_{kl} \geq 0$ and $\sum_{l=1}^{K} n_{kl} > 0$ $(k = 1, \ldots, K)$. Assume that the matrix $\{n_{kl}\}_{k,l=1,\ldots,K}$ is irreducible. A K-tuple of sets $(F_1, \ldots, F_K)$ is called a family of graph-directed sets for $\mathcal{S}$ if

$$F_k = \bigcup_{i=1}^{n_{k1}} S_i^{k1}(F_1) \cup \cdots \cup \bigcup_{i=1}^{n_{kK}} S_i^{kK}(F_K) \quad (k = 1, \ldots, K).$$

We have the following theorem.

Theorem 1. ([3], [5], [7], [8], [9]) *Let $K \geq 2$ and let $\mathcal{S}$ be as above. Then there is a unique K-tuple of non-empty compact graph-directed sets $(F_1, \ldots, F_K)$.*

We explain the random iteration algorithm with an example.

Example 2. Let T_i $(i = 1, 2, 3, 4)$ be a contraction, which is the similarity (dilation) that maps the whole square $\mathbf{X} = [0,1] \times [0,1]$ to the corresponding square in Fig. 2. Consider a pair of graph-directed sets (A, B) defined by

$$A = S_1^{11}(A) \cup S_1^{12}(B) \cup S_2^{12}(B),$$
$$B = S_1^{21}(A) \cup S_2^{21}(A) \cup S_1^{22}(B).$$

Here, each S_i^{kl} is defined as $S_1^{11} = T_2, S_1^{12} = T_1, S_2^{12} = T_4, S_1^{21} = T_1, S_2^{21} = T_4$ and $S_1^{22} = T_3$.

Let $x_1(0)$ and $x_2(0)$ be arbitrary points in $\mathbf{X}$ and choose randomly, recursively and independently

$$x_1(t+1) \in \{S_1^{11}(x_1(t)), S_1^{12}(x_2(t)), S_2^{12}(x_2(t))\},$$
$$x_2(t+1) \in \{S_1^{21}(x_1(t)), S_2^{21}(x_1(t)), S_1^{22}(x_2(t))\}.$$

The probabilities for selecting $\{S_1^{11}(x_1(t)), S_1^{12}(x_2(t)), S_2^{12}(x_2(t))\}$ as $x_1(t+1)$ and $\{S_1^{21}(x_1(t)), S_2^{21}(x_1(t)), S_1^{22}(x_2(t))\}$ as $x_2(t+1)$ are $(p_1^{11}, p_1^{12}, p_2^{12}) = (1/2, 1/4, 1/4)$ and $(p_1^{21}, p_2^{21}, p_1^{22}) = (1/4, 1/2, 1/4)$, respectively. Starting with $x_1(0) = (0,0)$ and $x_2(0) = (0,0)$, we obtained the pair of figures (A', B') in Fig. 2 after 10000 times loop.

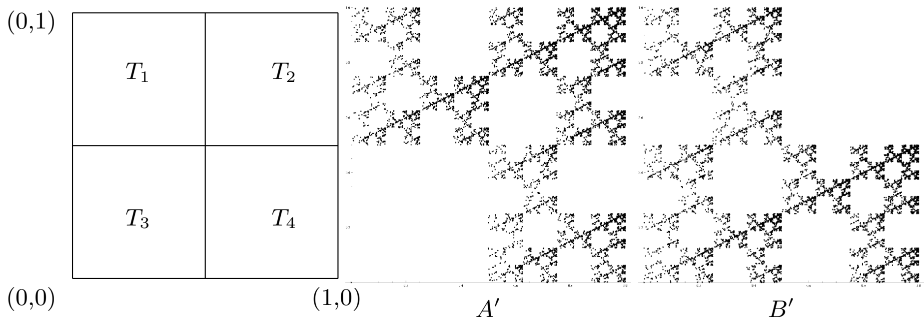

Fig. 2. An example of random iteration algorithm for graph-directed sets.

We will subsequently show that there is a unique pair of probability measures (μ_1, μ_2) on the pair of graph-directed sets (A, B) in Example 2 which satisfies

$$\mu_1 = p_1^{11}\ \mu_1 \circ (S_1^{11})^{-1} + \sum_{i=1}^{2} p_i^{12}\ \mu_2 \circ (S_i^{12})^{-1},$$

$$\mu_2 = \sum_{i=1}^{2} p_i^{21}\ \mu_1 \circ (S_i^{21})^{-1} + p_1^{22}\ \mu_2 \circ (S_1^{22})^{-1}.$$

For μ_1 and μ_2, it holds that for all $(x_1(0), x_2(0)) \in \mathbf{X} \times \mathbf{X}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} f(x_1(t)) = \int_{\mathbf{X}} f(x) d\mu_1(x),$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} f(x_2(t)) = \int_{\mathbf{X}} f(x) d\mu_2(x),$$

for almost all sequences $\{(x_1(t), x_2(t)) : t = 0, 1, \dots\}$, and for any continuous real function f on $\mathbf{X}$. In fact, for a unique probability measure $\tilde{\mu}$ on $\mathbf{X} \times \mathbf{X}$, it holds that for any $(x_1(0), x_2(0)) \in \mathbf{X} \times \mathbf{X}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} f(x_1(t), x_2(t)) = \int_{\mathbf{X}\times\mathbf{X}} f(x_1, x_2) d\tilde{\mu}(x_1, x_2) \quad \text{a.e.}$$

for any continuous real function f on $\mathbf{X} \times \mathbf{X}$. The measures μ_1 and μ_2 are the marginal distributions of the measure $\tilde{\mu}$ on $\mathbf{X} \times \mathbf{X}$.

Now, we state our *random iteration algorithm for a family of graph-directed sets*. Let $\mathbf{X}$ be a non-empty compact set in $\mathbf{R}^d$ such that $S_i^{kl}(\mathbf{X}) \subset \mathbf{X}$, for $k, l = 1, \dots, K, i = 1, \dots, n_{kl}$. A closed sphere $B(0, r)$ in $\mathbf{R}^d$ with a sufficiently large $r > 0$ such that $S_i^{kl}(B(0, r)) \subset B(0, r)$ for any k, l, i is an example of $\mathbf{X}$. For $k = 1, \dots, K$, let $(p_1^{k1}, \dots, p_{n_{k1}}^{k1}, \dots, p_1^{kK}, \dots, p_{n_{kK}}^{kK})$ be a system of probabilities assigned to $\{S_1^{k1},\dots,S_{n_{k1}}^{k1}\dots,S_1^{kK},\dots,S_{n_{kK}}^{kK}\}$, where $p_i^{kl} \geq 0$ $(i = 1,\dots,n_{k1})$ for $l = 1,\dots,K$ and $\sum_{l=1}^{K}\sum_{i=1}^{n_{kl}} p_i^{kl} = 1$.

Choose $(x_1(0),\dots,x_K(0)) \in \mathbf{X}^K$, and choose randomly, recursively and independently

$$x_k(t+1) \in \{S_i^{kl}(x_l(t)) : l = 1,\dots,K \text{ for which } n_{kl} > 0 \text{ and } i = 1,\dots,n_{kl}\},$$

for $k = 1,\dots,K$. The probability for the event $x_k(t+1) = S_i^{kl}(x_l(t))$ is p_i^{kl}. This produces a sequence of K-tuples of points $\{(x_1(t),\dots,x_K(t)) : t = 0,1,\dots\}$.

3 Invariant probability measure

Barnsley and Elton have shown the following.

Theorem 2. (Barnsley and Elton: [1], [2], [6]) *Let Y be a complete metric space. Let $\{T_1,\dots,T_N\}$ be a family of Lipschitz maps on Y. Let $(p_1,\dots,p_N)$ be a system of probabilities assigned to $\{T_1,\dots,T_N\}$, where $p_i > 0$ $(i = 1,\dots,N)$ and $\sum_{i=1}^{N} p_i = 1$. Suppose there exists $0 < r < 1$ such that*

$$\prod_{i=1}^{N} d(T_i(y),T_i(z))^{p_i} \leq r\ d(y,z)$$

for $y,z \in Y$.

Choose $y(0) \in Y$ and choose randomly, recursively and independently, $y(t) \in \{T_1(y(t-1)),\dots,T_N(y(t-1))\}$, where the probability for the event $\{y(t) = T_i(y(t-1))\}$ is p_i. Then the following hold.

(1) There is a unique invariant probability measure μ associated with transition probability $p(y,B) = \sum_{i=1}^{N} p_i 1_B(T_i(y))$, that is, $\mu(B) = \int p(y,B)d\mu(y)$ for all Borel set B.

(2) Let P be a probability $\prod_{i=1}^{\infty} P_i$ on $\prod_{i=1}^{\infty} J_i$, where $P_i = (p_1,\dots,p_N)$ and $J_i = \{1,\dots,N\}$. It holds that for any $y(0) \in Y$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1} f(y(t)) = \int_Y f(y)d\mu(y)\ P-\text{a.e.}$$

for all continuous function $f : Y \to \mathbf{R}$.

Let us note that μ is an invariant probability measure if and only if $\mu = M(\mu)$ for the Markov operator

$$M(\nu) = \sum_{i=1}^{N} p_i\,\nu\circ T_i^{-1}.$$

By applying Barnsley and Elton's theorem, we show the uniqueness of an invariant probability measure of a random iteration algorithm for a family of graph-directed sets. Recall that $\mathbf{X}$ is a non-empty compact set in $\mathbf{R}^d$ such that $S_i^{kl}(\mathbf{X}) \subset \mathbf{X}$ for $k, l = 1, \ldots, K, i = 1, \ldots, n_{kl}$. Put $\mathbf{X}_k = \mathbf{X}$ for $k = 1, \ldots, K$, and define $\mathbf{X}^K = \mathbf{X}_1 \times \cdots \mathbf{X}_K$. Define a metric d on $\mathbf{X}^K$ by

$$d((x_1, \ldots, x_K), (y_1, \ldots, y_K)) = \mathrm{Max}\{|x_k - y_k| : k \ = 1, \ldots, K\},$$

where $|x_k - y_k|$ denotes the d-dimensional Euclidean metric.

Put $I_k = \{(l_k, i_k) : n_{kl_k} > 0,\ 1 \leq i_k \leq n_{kl_k}\} \subset \{1, \ldots, K\} \times \mathbf{N}$ for $k = 1, \ldots, K$. Put further $I = I_1 \times \cdots \times I_K$. For $S_i^{kl} : \mathbf{X} \to \mathbf{X}$, where $k = 1, \ldots, K$ and $(l, i) \in I_k$, let $\tilde{S}_i^{kl} : \mathbf{X}^K \to \mathbf{X}_k$ be defined by $\tilde{S}_i^{kl}(x_1, \ldots, x_K) = S_i^{kl}(x_l)$.

For $((l_1, i_1), \ldots, (l_K, i_K)) \in I$, a transformation $T_{((l_1,i_1),\ldots,(l_K,i_K))} : \mathbf{X}^K \to \mathbf{X}^K$ is defined by

$$\begin{aligned} T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1, \ldots, x_K) &:= (\tilde{S}_{i_1}^{1l_1}(x_1, \ldots, x_K), \ldots, \tilde{S}_{i_K}^{Kl_K}(x_1, \ldots, x_K)) \\ &= (S_{i_1}^{1l_1}(x_{l_1}), \ldots, S_{i_K}^{Kl_K}(x_{l_K})) \end{aligned}$$

with the associated probability

$$p_{((l_1,i_1),\ldots,(l_K,i_K))} = p_{i_1}^{1l_1} \cdots p_{i_K}^{Kl_K}.$$

We apply Barnsley and Elton's theorem to $Y = \mathbf{X}^K$ and

$$\mathcal{T} = \{T_{((l_1,i_1),\ldots,(l_K,i_K))} :\ ((l_1, i_1), \ldots, (l_K, i_K)) \in I\}$$

with probabilities $p_{i_1}^{1l_1} \cdots p_{i_K}^{Kl_K}$. Let L be the set of functions as defined below.

$$\begin{aligned} L = \{ & f : \mathbf{X}^K \to \mathbf{R} : \\ & |f(x_1, \ldots, x_K) - f(y_1, \ldots, y_K)| \leq \mathrm{Max}\{|x_k - y_k| : k \ = 1, \ldots, K\}\}, \end{aligned}$$

where $|x_k - y_k|$ denotes the d-dimensional Euclidean metric.

Let $\mathbf{P}(\mathbf{X}^K)$ be the space of normalized Borel measures on $\mathbf{X}^K$. The Hutchinson metric d_H of $\mathbf{P}(\mathbf{X}^K)$ is defined by

$$d_H(\mu, \nu) = \mathrm{Sup}\Big\{ \int f d\mu - \int f d\nu : f \in L \Big\}.$$

It is well known that $(\mathbf{P}(\mathbf{X}^K), d_H)$ is a compact space. (See Barnsley [1].)

Let us define a Markov operator $M : \mathbf{P}(\mathbf{X}^K) \to \mathbf{P}(\mathbf{X}^K)$, and prove a theorem which claims the existence of a certain measure.

Definition 3. *The Markov operator associated with*

$$\mathcal{T} = \{T_{((l_1,i_1),\ldots,(l_1,i_1))} : ((l_1, i_1), \ldots, (l_K, i_K)) \in I\}$$

is a transformation $M : \mathbf{P}(\mathbf{X}^K) \to \mathbf{P}(\mathbf{X}^K)$ *defined by*

$$M(\nu) = \sum_{((l_1,i_1),\ldots,(l_K,i_K)) \in I} \prod_{k=1}^{K} p_{i_k}^{kl_k} \nu \circ (T_{((l_1,i_1),\ldots,(l_K,i_K))})^{-1}.$$

Theorem 3. *There exists a unique probability measure $\tilde{\mu}$ on $\mathbf{X}^K$ such that $\tilde{\mu} = M(\tilde{\mu})$.*

Proof (Proof1: Application of Barnsley and Elton's criterion). Recall that, for $((l_1, i_1), \ldots, (l_K, i_K)) \in I$,

$$T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1, \ldots, x_K) = (S_{i_1}^{1l_1}(x_{l_1}), \ldots, S_{i_K}^{Kl_K}(x_{l_K})).$$

Let s be the maximum of the contraction ratios of $\{S_i^{kl}\}$. Note that $s < 1$. Recall that $d((x_1, \ldots, x_K), (y_1, \ldots, y_K)) = \mathrm{Max}\{|x_k - y_k| : k = 1, \ldots, K\}$, where $|x_k - y_k|$ denotes the d-dimensional Euclidean metric. Then it holds that

$$\begin{aligned}
&d(T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1, \ldots, x_K)), T_{((l_1,i_1),\ldots,(l_K,i_K))}(y_1, \ldots, y_K))) \\
&\quad = d((S_{i_1}^{1l_1}(x_{l_1}), \ldots, S_{i_K}^{Kl_K}(x_{l_K})), (S_{i_1}^{1l_1}(y_{l_1}), \ldots, S_{i_K}^{Kl_K}(y_{l_K}))) \\
&\quad = \mathrm{Max}\{|S_{i_1}^{1l_1}(x_{l_1}) - S_{i_1}^{1l_1}(y_{l_1})|, \ldots, |S_{i_K}^{Kl_K}(x_{l_K}) - S_{i_K}^{Kl_K}(y_{l_K})|\} \\
&\quad \leq s\mathrm{Max}\{|x_{l_1} - y_{l_1}|, \ldots, |x_{l_K} - y_{l_K}|\} \\
&\quad \leq s\mathrm{Max}\{|x_1 - y_1|, \ldots, |x_K - y_K|\}. \qquad (1)
\end{aligned}$$

The Barnsley and Elton's condition holds if $d(T_i(x), T_i(y)) \leq sd(x, y)$ for an $s < 1$. From (1) above this criterion is satisfied, and so we can apply the Barnsley and Elton's theorem and obtain the desired measure. □

Proof (Proof2: Direct proof). Notice that, for $f \in L$,

$$\begin{aligned}
&\left| f(T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1, \ldots, x_K)) - f(T_{((l_1,i_1),\ldots,(l_K,i_K))}(y_1, \ldots, y_K)) \right| \\
&\quad = \left| f(S_{i_1}^{1l_1}(x_{l_1}), \ldots, S_{i_K}^{Kl_K}(x_{l_K})) - f(S_{i_1}^{1l_1}(y_{l_1}), \ldots, S_{i_K}^{Kl_K}(y_{l_K})) \right| \\
&\quad \leq \mathrm{Max}\{|S_{i_1}^{1l_1}(x_{l_1}) - S_{i_1}^{1l_1}(y_{l_1})|, \ldots, |S_{i_K}^{Kl_K}(x_{l_K}) - S_{i_K}^{Kl_K}(y_{l_K})|\} \\
&\quad \leq s\mathrm{Max}\{|x_{l_1} - y_{l_1}|, \ldots, |x_{l_K} - y_{l_K}|\} \\
&\quad \leq s\mathrm{Max}\{|x_1 - y_1|, \ldots, |x_K - y_K|\}.
\end{aligned}$$

Define

$$\hat{f}(x_1, \ldots, x_K) = s^{-1} \sum_{((l_1,i_1),\ldots,(l_K,i_K)) \in I} \prod_{k=1}^{K} p_{i_k}^{kl_k} f(T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1, \ldots, x_K)).$$

Then

$$\begin{aligned}
&\left| \hat{f}(x_1, \ldots, x_K) - \hat{f}(y_1, \ldots, y_K) \right| \\
&\quad \leq s^{-1} \sum_{((l_1,i_1),\ldots,(l_K,i_K)) \in I} \prod_{k=1}^{K} p_{i_k}^{kl_k} s \ \mathrm{Max}\{|x_1 - y_1|, \ldots, |x_K - y_K|\} \\
&\quad \leq \mathrm{Max}\{|x_1 - y_1|, \ldots, |x_K - y_K|\},
\end{aligned}$$

since $\sum_{((l_1,i_1),\ldots,(l_K,i_K))\in I}\prod_{k=1}^{K}p_{i_k}^{kl_k}=1$. It therefore follows that $\hat{f}\in L$. If we put $\hat{L}=\{\hat{f}(x_1,\ldots,x_K): f\in L\}$, then $\hat{L}\subset L$ holds.

By the definition,

$$\begin{aligned}
d_H(M(\mu),M(\nu)) &= \mathrm{Sup}\Big\{\int f dM(\mu)-\int f dM(\nu): f\in L\Big\}\\
&= \mathrm{Sup}\Big\{\int \sum_{((l_1,i_1),\ldots,(l_K,i_K))\in I}\prod_{k=1}^{K}p_{i_k}^{kl_k}\\
&\qquad f(T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1,\ldots,x_K))d\mu(x_1,\ldots,x_K)\\
&\qquad -\int \sum_{((l_1,i_1),\ldots,(l_K,i_K))\in I}\prod_{k=1}^{K}p_{i_k}^{kl_k}\\
&\qquad f(T_{((l_1,i_1),\ldots,(l_K,i_K))}(x_1,\ldots,x_K)d\nu(x_1,\ldots,x_K): f\in L\Big\}\\
&= \mathrm{Sup}\Big\{s\Big(\int \hat{f}(x_1,\ldots,x_K)d\mu(x_1,\ldots,x_K)\\
&\qquad\qquad -\int \hat{f}(x_1,\ldots,x_K)d\nu(x_1,\ldots,x_K)\Big): \hat{f}\in\hat{L}\Big\}\\
&\le \mathrm{Sup}\Big\{s\Big(\int f(x_1,\ldots,x_K)d\mu(x_1,\ldots,x_K)\\
&\qquad\qquad -\int f(x_1,\ldots,x_K)d\nu(x_1,\ldots,x_K)\Big): f\in L\Big\}\\
&= s\ d_H(\mu,\nu).
\end{aligned}$$

Therefore the Markov operator M is a contraction map on $\mathbf{P}(\mathbf{X}^K)$. This implies that there is a unique invariant probability measure $\tilde{\mu}$ in $\mathbf{P}(\mathbf{X}^K)$. □

Barnsley and Elton's theorem for random iterated algorithms can be extended to a family of graph-directed sets.

Theorem 4. *Let $\tilde{\mu}$ be the unique invariant probability measure claimed in Theorem 3. Then for any $(x_1(0),\ldots,x_K(0))\in\mathbf{X}^K$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1}f(x_1(t),\ldots,x_K(t))=\int_{\mathbf{X}^K}f(x_1,\ldots,x_K)d\tilde{\mu}(x_1,\ldots,x_K)\quad \text{a.e.}$$

for all continuous function $f:\mathbf{X}^K\to\mathbf{R}$.

Proof. We apply (2) of Barnsley and Elton's theorem to $T_{((l_1,i_1),\ldots,(l_K,i_K))}$ on $\mathbf{X}^K$ with probabilities $\prod_{k=1}^{K}p_{i_k}^{kl_k}$. □

Corollary 1. *(1) For the marginal distributions $\tilde{\mu}_1,\ldots,\tilde{\mu}_K$, it holds that*

$$\tilde{\mu}_k=\sum_{l=1}^{K}\sum_{i=1}^{n_{kl}}p_i^{kl}\tilde{\mu}_l\circ(S_i^{kl})^{-1}$$

for $k=1,\ldots,K$.

(2) For any $(x_1(0), \ldots, x_K(0)) \in \mathbf{X}^K$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} g(x_k(t)) = \int_{\mathbf{X}} g(x) d\tilde{\mu}_k(x) \quad \text{a.e.}$$

for all continuous function $g : \mathbf{X} \to \mathbf{R}$ *and for* $k = 1, \ldots, K$.

Proof. Proof of (1). Note that for a family of Borel sets $A_1, \ldots, A_K$ in $\mathbf{X}$, it holds that

$$\begin{aligned}
&(T_{((l_1,i_1),\ldots,(l_K,i_K))})^{-1}(A_1 \times \cdots \times A_K) \\
&\qquad = \{(x_1, \ldots, x_K) \ : \ \tilde{S}^{kl_k}_{i_k}(x_1, \ldots, x_K) \in A_k, k = 1, \ldots, K\} \\
&\qquad = \bigcap_{k=1}^{K} (\tilde{S}^{kl_k}_{i_k})^{-1}(A_k).
\end{aligned}$$

So we have

$$(T_{((l_1,i_1),\ldots,(l_K,i_K))})^{-1}(\mathbf{X}_1 \times \cdots \times \mathbf{X}_{k-1} \times A_k \times \mathbf{X}_{k+1} \cdots \times \mathbf{X}_K) = (\tilde{S}^{kl_k}_{i_k})^{-1}(A_k),$$

because $(\tilde{S}^{jl_j}_{i_j})^{-1}(\mathbf{X}_j) = \mathbf{X}^K$. Recall that $\mathbf{X}_l = \mathbf{X}$ for all l. Note that $\tilde{\mu} = M(\tilde{\mu})$. Then it holds that

$$\begin{aligned}
\tilde{\mu}_k(A) &= \tilde{\mu}(\mathbf{X}_1 \times \cdots \times \mathbf{X}_{k-1} \times A \times \mathbf{X}_{k+1} \cdots \times \mathbf{X}_K) \\
&= M(\tilde{\mu})(\mathbf{X}_1 \times \cdots \times \mathbf{X}_{k-1} \times A \times \mathbf{X}_{k+1} \cdots \times \mathbf{X}_K) \\
&= \sum_{((l_1,i_1),\ldots,(l_K,i_K)) \in I} \prod_{j=1}^{K} p^{jl_j}_{i_j} \\
&\qquad \tilde{\mu}((T_{((l_1,i_1),\ldots,(l_K,i_K))})^{-1}(\mathbf{X}_1 \times \cdots \times \mathbf{X}_{k-1} \times A \times \mathbf{X}_{k+1} \cdots \times \mathbf{X}_K)) \\
&= \sum_{((l_1,i_1),\ldots,(l_K,i_K)) \in I} \prod_{j=1}^{K} p^{jl_j}_{i_j} \tilde{\mu}((\tilde{S}^{kl_k}_{i_k})^{-1}(A)) \\
&= \sum_{(l_k,i_k) \in I_k} p^{kl_k}_{i_k} \tilde{\mu}((\tilde{S}^{kl_k}_{i_k})^{-1}(A)) \prod_{j \neq k} \sum_{(l_j,i_j) \in I_j} p^{jl_j}_{i_j} \\
&= \sum_{(l_k,i_k) \in I_k} p^{kl_k}_{i_k} \tilde{\mu}((\tilde{S}^{kl_k}_{i_k})^{-1}(A)) \\
&= \sum_{(l_k,i_k) \in I_k} p^{kl_k}_{i_k} \tilde{\mu}_{l_k}((\tilde{S}^{kl_k}_{i_k})^{-1}(A)).
\end{aligned}$$

This proves the assertion (1).

Proof of (2). Define $f(x_1, \ldots, x_K) = g(x_k)$. Then by virtue of Theorem 4, it holds that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1} g(x_k(t)) = \lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1} f(x_1(t),\dots,x_K(t))$$
$$= \int_{\mathbf{X}^K} f(x_1,\dots,x_K)d\tilde{\mu}(x_1,\dots,x_K) \quad \text{a.e.}$$
$$= \int_{\mathbf{X}} g(x)d\tilde{\mu}_k(x).$$

We thus have

$$\lim_{n\to\infty}\tfrac{1}{n}\textstyle\sum_{t=0}^{n-1} g(x_k(t)) = \int_{\mathbf{X}} g(x)d\tilde{\mu}_k(x) \quad \text{a.e.}$$

for all continuous function $g : \mathbf{X} \to \mathbf{R}$ and k=1, ..., K.

This proves the assertion (2).

□

Theorem 5. *Let $\tilde{\mu}$ be the unique probability measure in Theorem 3, and let $\tilde{\mu}_1,\dots,\tilde{\mu}_K$ be the marginal distributions of $\tilde{\mu}$. Then for $m = 1,\dots,K$, the support of $\tilde{\mu}_m$ is F_m, where $(F_1,\dots,F_K)$ is the family of graph-directed sets in Theorem 1.*

Proof. The proof is analogous to that of Theorem 2 in Section 9.6 of [1].

Let A denote the support of $\tilde{\mu}$. Notice that

$$T_{((l_1,i_1),\dots,(l_K,i_K))}(F_1\times\cdots\times F_K) \subset F_1\times\cdots\times F_K$$

for any $((l_1,i_1),\dots,(l_K,i_K)) \in I$. It follows that $\{T_{((l_1,i_1),\dots,(l_K,i_K))}\}$ restricted on $F_1\times\cdots\times F_K$ defines a random iteration algorithm with the probabilities $\prod_{k=1}^{K} p_{i_k}^{kl_k}$. Let $\tilde{\nu}$ be an invariant probability measure for the restricted random iteration algorithm, and this $\tilde{\nu}$ is an invariant probability measure for the random iteration algorithm on $\mathbf{X}^K$. Since $\tilde{\mu}$ is unique, $\tilde{\mu} = \tilde{\nu}$. It follows that $A \subset F_1 \times \cdots \times F_K$, and so the support of $\tilde{\mu}_m$ is included in F_m.

For $m = 1,\dots,K$, let Σ_m be the set of sequences $\{(l_1,i_1;\dots;l_n,i_n;\dots) : n_{l_{n-1}\, l_n} > 0,\ 1 \le i_n \le n_{l_{n-1}\, l_n} \ \text{for}\ \ n = 1,\dots\}$, where $l_0 = m$.

For each point $a \in F_m$, there is a (not necessarily unique) sequence in Σ_m such that

$$a \in S_{i_1}^{ml_1} \circ S_{i_2}^{l_1 l_2} \circ \cdots \circ S_{i_n}^{l_{n-1}l_n}(\mathbf{X}_{l_n})$$

holds for all n. Let O be an open set in $\mathbf{X}$ which contains a. By the fact that S_i^{kl} is a contraction, there is a positive integer n such that

$$S_{i_1}^{ml_1} \circ S_{i_2}^{l_1 l_2} \circ \cdots \circ S_{i_n}^{l_{n-1}l_n}(\mathbf{X}_{l_n})) \subset O.$$

Note that $\tilde{\mu}_m(S_{i_1}^{ml_1} \circ S_{i_2}^{l_1 l_2} \circ \cdots \circ S_{i_n}^{l_{n-1}l_n}(\mathbf{X}_{l_n})) \ge \prod_{j=1}^{n} p_{i_j}^{l_{j-1}l_j} > 0$. It holds that $\tilde{\mu}_m(O) > 0$, and so F_m is included in the support of $\tilde{\mu}_m$. □

Remark 1. In the above proofs we have not used the independence of choosing $\{S_{i_1}^{1l_1}, \ldots, S_{i_K}^{Kl_K}\}$, or the productivity of the probabilities $\prod_{k=1}^{K} p_{i_k}^{kl_k}$. So we can formulate the random iteration algorithm so that the probability of choosing $\{S_{i_1}^{1l_1}, \ldots, S_{i_K}^{Kl_K}\}$ can be expressed as $p_{(l_1,i_1;\ldots,l_K,i_K)}$, which is not restricted to the independent case of $p_{i_1}^{1l_1} \ldots p_{i_K}^{Kl_K}$. Theorems 3, 4 and 5 hold for thus modified random iteration algorithm.

Remark 2. We propose a variation of this algorithm which changes only one coordinate X_k on each step. Let $\{q_1, \ldots, q_K\}$ be a probability, that is, $q_k > 0$ for $k = 1, \ldots, K$ and $\sum_{k=1}^{K} q_k = 1$. For $k = 1, \ldots, K$, let $(p_1^{k1}, \ldots, p_{n_{k1}}^{k1}, \ldots, p_1^{kK}, \ldots, p_{n_{kK}}^{kK})$ be a system of probabilities defined in Section 2.

Choose $(x_1(0), \ldots, x_K(0)) \in \mathbf{X}^K$. Next choose randomly $k(1) \in \{1, \ldots, K\}$, with probability $q_{k(1)}$, and then choose randomly $S_i^{k(1)l}(x_l(0))$ for $l = 1, \ldots, K$ with $n_{k(1)l} > 0$ and $1 \leq i \leq n_{k(1)l}$, with probability $p_i^{k(1)l}$. Let $x_{k(1)}(1) = S_i^{k(1)l}(x_l(0))$ and $x_j(1) = x_j(0)$ for $j \neq k(1)$. Continue this procedure recursively and independently.

So we have

$$\begin{aligned} x_{k(t+1)}(t+1) &= S_i^{k(t+1)l}(x_l(t)), \\ x_j(t+1) &= x_j(t) \text{ for } j \neq k(t+1), \end{aligned}$$

with probability $q_{k(t+1)} p_i^{k(t+1)l}$, where $k(t+1) = 1, \ldots, K$, $l = 1, \ldots, K$ with $n_{k(t+1)l} > 0$ and $1 \leq i \leq n_{k(t+1)l}$.

This produces a sequence of K-tuples of points $\{(x_1(t), \ldots, x_K(t)) : t = 0, 1, \ldots\}$. We then have the following results.

(1) There exists a unique probability measure $\hat{\mu}$ on $\mathbf{X}^K$ such that $\hat{\mu} = \hat{M}(\hat{\mu})$, where $\hat{M}$ is the associated Markov operator.

(2) Let $\hat{\mu}_1, \ldots, \hat{\mu}_K$ be the marginal distributions of $\hat{\mu}$. Then for $m = 1, \ldots, K$, the support of $\hat{\mu}_m$ is F_m, where $(F_1, \ldots, F_K)$ is the family of graph-directed sets in Theorem 1.

(3) For any $(x_1(0), \ldots, x_K(0)) \in \mathbf{X}^K$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} f(x_1(t), \ldots, x_K(t)) = \int_{\mathbf{X}^K} f(x_1, \ldots, x_K) d\hat{\mu}(x_1, \ldots, x_K) \quad \text{a.e.}$$

for all continuous function $f : \mathbf{X}^K \to \mathbf{R}$.

(4) (i) For the marginal distributions $\hat{\mu}_1, \ldots, \hat{\mu}_K$, it holds that

$$\hat{\mu}_k = \sum_{l=1}^{K} \sum_{i=1}^{n_{kl}} p_i^{kl} \hat{\mu}_l \circ (S_i^{kl})^{-1}$$

for $k = 1, \ldots, K$.

(ii) For any $(x_1(0), \ldots, x_K(0)) \in \mathbf{X}^K$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} g(x_k(t)) = \int_{\mathbf{X}} g(x) d\hat{\mu}_k(x) \quad \text{a.e.}$$

for all continuous function $g : \mathbf{X} \to \mathbf{R}$ and for $k = 1, \ldots, K$.

Acknowledgement This work has been supported in part by Research Grant from KSU(2008, 282), Research Grant from KSU(2008, 339), JSPS Grant-in-Aid No. 20540143, and JSPS Grant-in-Aid No. 18500013, respectively.

References

1. M. Barnsley, Fractals everywhere. *Academic Press, INC*, Boston, 2000.
2. M. Barnsley and J. Elton, A new class of Markov processes for image encoding. *Adv. Appl. Prob.*, 20, 14-32, 1988.
3. T. Bedford, Dimension and dynamics of fractal recurrent sets. *J. London Math. Soc. (2)*, 33:89-100, 1986.
4. V. Brattka, Some Notes on Fine Computability. *Journal of Universal Computer Science*, 8:382-395, 2002.
5. G.A. Edgar, Measure, Topology and Fractal Geometry. *Springer-Verlag*, 1990.
6. J.H. Elton, An ergodic theorem for iterated maps. *Ergodic Theory and Dynamical Systems*, 7:481-488, 1987.
7. K. J. Falconer, Fractal Geometry. *John Wiley & Sons*, Chichester, 1990.
8. J. E. Hutchinson, Fractals and self similarity. *Indiana Univ. Math. J.*,30:713-747, 1981.
9. R.D. Mauldin and S.C. Williams, Hausdorff dimension in graph directed constructions. *Trans. Amer. math. Soc*,309:811-829, 1988.
10. Y. Tsujii, T.Mori, Y.Yasugi and H.Tsuiki, Fine-Continuous Functions and Fractals Defined by Infinite Systems of Contractions. Proceeding Volume ILC2007 (to appear).
11. I.Werner, Contractive Markov Systems. J. London Math. Soc. (2) 71:236-258, 2005.